Pair of Straight lines

Homogeneous equation

$ax^3 + 2hx^2y + 2gxy^2 + by^3 = 0 \longrightarrow$ Homogeneous cubic eqn.

$ax^2 + 2hxy + by^2 = 0 \longrightarrow$ Homogeneous quadratic eqn.

Non-Homogeneous eqn.

(i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

(ii) $x^3 + 2x^2y + 3y^2 + c = 0$

(iii) $x^2 + y^2 - 5 = 0$ (iv) $ax + by + c = 0$

Art. 36: Homogeneous quadratic equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines passing through the origin.

We have the eqn. $ax^2 + 2hxy + by^2 = 0$

Dividing both sides by $x^2 b$, $(b \neq 0)$

$$\frac{a}{b} + \frac{2hxy}{x^2 b} + \frac{y^2}{x^2} = 0$$

$$\left(\frac{y}{x}\right)^2 + \frac{2h}{b} \cdot \frac{y}{x} + \frac{a}{b} = 0 \quad \text{—— (i)}$$

Eqn. (i) is a quadratic in $\frac{y}{x}$; it has two roots; suppose m_1 & m_2.

$$\therefore m_1 + m_2 = \frac{-2h/b}{1} = -\frac{2h}{b}$$

$$m_1 \cdot m_2 = \frac{a/b}{1} = \frac{a}{b}$$

$$\therefore \left(\frac{y}{x}\right)^2 + \frac{2h}{b} \cdot \frac{y}{x} + \frac{a}{b} \equiv \left(\frac{y}{x} - m_1\right)\left(\frac{y}{x} - m_2\right) = 0$$

i.e. $\frac{y}{x} - m_1 = 0$ & $\frac{y}{x} - m_2 = 0$

$ax^2 + bx + c = 0$; if two roots α, β, then $\alpha + \beta = -b/a$ & $\alpha\beta = \frac{c}{a}$

$ax^2 + bx + c = 0$, roots α, β
then $(x - \alpha)(x - \beta) = 0$

Exp: $x^2 - 5x + 6 = 0$
$(x-3)(x-2) = 0$
$\therefore x = 3, 2$

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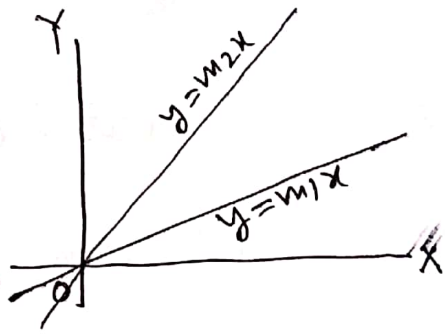
$y - m_1 x = 0$ & $y - m_2 x = 0$

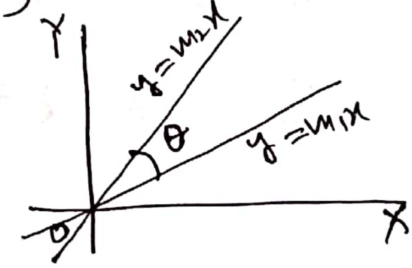
$y = m_1 x$ & $y = m_2 x$, which represents a pair of st. lines passing through the origin.



Art 38: Angle between two lines represented by $ax^2 + 2hxy + by^2 = 0$

We have $ax^2 + 2hxy + by^2 \equiv (y - m_1 x)(y - m_2 x) = 0$

$\therefore m_1 + m_2 = -\frac{2h}{b}$; $m_1 m_2 = \frac{a}{b}$



If θ be the angle between the two lines, then

$y - m_1 x = 0$,
$y - m_2 x = 0$
Angle betn two lines

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$= \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}$$

$$= \frac{\sqrt{\left(-\frac{2h}{b}\right)^2 - 4\cdot\frac{a}{b}}}{1 + \frac{a}{b}}$$

$$= \frac{\sqrt{\frac{4h^2}{b^2} - 4\frac{a}{b}}}{1 + \frac{a}{b}}$$

$$= \frac{2\sqrt{h^2 - ab}}{b + a}$$

$$\therefore \theta = \tan^{-1}\left(\frac{2\sqrt{h^2 - ab}}{a + b}\right).$$

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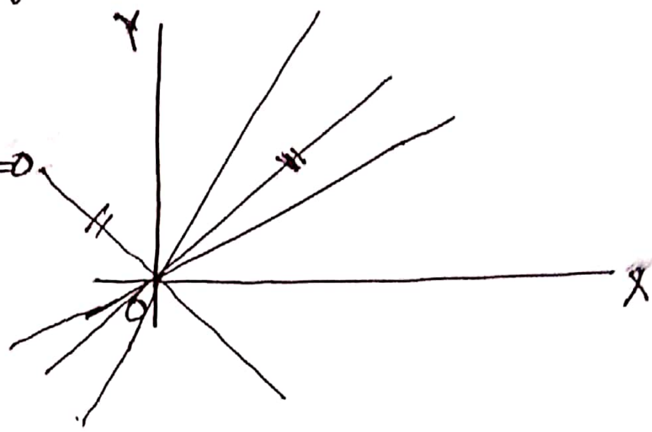
Art. 39: Equation of bisectors of the angles between the lines represented by $ax^2+2hxy+by^2=0$.



We have the lines ~~ax^2+~~

$$ax^2+2hxy+by^2 \equiv (y-m_1x)(y-m_2x)=0$$

$$\therefore\ y-m_1x=0 \ \& \ y-m_2x=0$$

The equ. of bisectors of the angles between them;

$$\frac{y-m_1x}{\sqrt{1+m_1^2}} = \pm\frac{y-m_2x}{\sqrt{1+m_2^2}}$$

or, $(y-m_1x)^2(1+m_2^2) = (y-m_2x)^2(1+m_1^2)$

or, $(y^2-2m_1xy+m_1^2x^2)(1+m_2^2) = (y^2-2xym_2+m_2^2x^2)(1+m_1^2)$

or, $\cancel{y^2}-2m_1xy+m_1^2x^2+m_2^2y^2-2m_1m_2^2xy+\cancel{m_1^2m_2^2x^2}$
$= \cancel{y^2}-2m_2xy+m_2^2x^2+m_1^2y^2-2m_1^2m_2xy+\cancel{m_1^2m_2^2x^2}$

or, $(m_1^2-m_2^2)x^2+(m_2^2-m_1^2)y^2 = 2(m_1-m_2)xy$
$+2m_1m_2(m_2-m_1)xy$

or, $(m_1^2-m_2^2)(x^2-y^2) = 2xy\{(m_1-m_2)-(m_1-m_2)m_1m_2\}$

or, $(m_1+m_2)(x^2-y^2) = 2xy\{1-m_1m_2\}$

or, $-\frac{\cancel{2}h}{b}\cdot(x^2-y^2) = \cancel{2}xy\left(1-\frac{a}{b}\right)$

or, $\frac{h(x^2-y^2)}{-b} = \frac{(b-a)xy}{b}$

or, $\frac{h(x^2-y^2)}{-\cancel{b}} = \frac{(a-b)xy}{-\cancel{b}}$

or, $\frac{x^2-y^2}{a-b} = \frac{xy}{h}$; which are the equations of bisectors.

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Q. Find the condition that the general equation of 2nd degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of st. lines.

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{———} (1)$$

If we transfer the origin to the point (α, β), then

(1) becomes $a(x+\alpha)^2+2h(x+\alpha)(y+\beta)+b(y+\beta)^2+2g(x+\alpha)$
$+2f(y+\beta)+c=0$

$$ax^2+2a\alpha x+a\alpha^2+2h(xy+\alpha y+\beta x+\alpha\beta)+by^2$$
$$+2b\beta y+b\beta^2+2gx+2g\alpha+2fy+2f\beta+c=0$$

$$ax^2+2hxy+by^2+2(a\alpha+h\beta+g)x+2(h\alpha+b\beta+f)y$$
$$+a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c=0 \quad \text{———} (2)$$

Equation (2) will represent a pair of st. lines if

$$a\alpha+h\beta+g=0 \quad \text{———} (3)$$
$$h\alpha+b\beta+f=0 \quad \text{———} (4)$$
& $$a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c=0 \quad \text{———} (5)$$

(5) $\Rightarrow$ $\alpha(a\alpha+h\beta+g)+\beta(h\alpha+b\beta+f)+g\alpha+f\beta+c=0$

or, $\alpha\cdot 0+\beta\cdot 0+g\alpha+f\beta+c=0$ [by (3) & (4)]

or, $g\alpha+f\beta+c=0 \quad \text{———} (6)$

If we eliminate α, β from the eqns. (3), (4) & (6)

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

or, $abc+2fgh-af^2-bg^2-ch^2=0$ which is the required condition.

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